Divisors of positive integers

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May 2021

in mathematics,

There is a theorem, which is proved on the basis of certain axioms. We will start with the fundamental theorem of arithmetic. This theorem applies to positive integers, those are whole numbers one and greater. Rational numbers and other non-integer real numbers will not be considered here. This is number theory. Let b be an arbitrary chosen positive integer. In words, the fundamental theorem of arithmetic states that every positive integer has a unique prime factorization. Now prime numbers are only divisible by one and themselves. The set of all prime numbers goes like this –

P = {2, 3, 5, 7, 11, …} (expression 1)

The fundamental theorem of arithmetic goes like this

$b = p0^{e0}p1^{e1}p2^{e2}$ … (expression 2)

where the pi are prime numbers, and the ej are positive integer exponents.

For example $18 = 2*3^2$.

Here is another example 30 = 2*3*5.

And if we choose to put the prime numbers from smallest to largest, and not put any one as factors, then there is only one unique of factorization for every integer. This is standard mathematics I learned in school.

More Preliminaries

Now there is a concept of discrete divisors. By definition, the list of discrete divisors of a positive integer b includes 1. The greatest discrete divisor of a positive integer b is less than or equal to b.

Using a wonderful computer algebra system called Maple, I was able to do some simple computer coding to make a Discrete_Divisors(b) procedure. I worked many examples to try to get a handle on the behavior of this Discrete_Divisors(b) procedure

Now assume b is a prime, or b is a semi-prime, that is b = p*q where p and q are both prime numbers. If b is a prime number then the count of its Discrete Divisors is 1 and its only discrete divisor is 1. This is inferred by observation and is not a theorem from my point of view.

Similarly,

Also, if b is the product of d distinct primes, each with exponent 1, then the count of the discrete divisors will be d. In other words, assume

b = p1*p2* … * pd. (expression 3)

Then b has exactly d discrete divisors. Again, by observation.

Next form of b.

Assume b has one repeated prime factor. (repeated ‘a’ times) So

b = $p0^a$.

written with less pretty typesetting, we have

b = p0^a. (expression 4)

For example, choose p0=3 and let a = 1,2,3, … c.

Here is our data table

‘a’	count_of_discrete_divisors(3^a).
1	1
2	2
3	3
4	4

Numerical evidence shows that a number that is a single prime power with exponent ‘a’, will have exactly ‘a’ different discrete divisors. That is, our number b has one repeated prime factor only.

Appendix 1

Divisors Procedure in Maple

```
>

> Divisors := proc (n) local d, count; description " Enumerate all proper divisors.  Assume a positive integer input.     Then count the proper divisors."; print(" Input is ", n, " Begin calculation."); count := 0; for d to (1/2)*n do if `mod`(n, d) = 0 then count := count+1; print("   One proper divisor of ", n, " is the number ", d) end if end do; print(" and that is all of them.  "); print(" count of proper divisors is ", count) end proc;


> Describe(Divisors);

%;


#  Enumerate all proper divisors. Assume a positive integer input. Then count

# the proper divisors.

Divisors( n )


# example for Proper_Divisors(4)

> Divisors(2^2);

               " Input is ", 4, " Begin calculation."

          "   One proper divisor of ", 4, " is the number ", 1

          "   One proper divisor of ", 4, " is the number ", 2

                   " and that is all of them.  "

                 " count of proper divisors is ", 2

# good fun
```

End Appendix 1

Appendix 2

Examples of Discrete Prime Divisors procedure to demonstrate that square free positive integers with exactly d prime divisors have exactly d discrete prime divisors. In other words, assume

b = p1*p2* … * pd.

See examples ~

Proper divisors of a prime number has only one as its prime divisor.

```
> ProperDivisors(19);

          " Input is ", 19, " Begin calculation."

      "   One proper divisor of ", 19, " is the number ", 1

             " and that is all of them.  "

           " count of proper divisors is ", 1


> ProperDivisors(17*(11*(5*7)*13));
```

so 85085 = 5*7*11*13*17 has 5 distinct prime divisors and is square free. (prime divisors are without repetition.)

```
         " Input is ", 85085, " Begin calculation."

     "   One proper divisor of ", 85085, " is the number ", 1

     "   One proper divisor of ", 85085, " is the number ", 5

     "   One proper divisor of ", 85085, " is the number ", 7

     "   One proper divisor of ", 85085, " is the number ", 11

     "   One proper divisor of ", 85085, " is the number ", 13

     "   One proper divisor of ", 85085, " is the number ", 17

     "   One proper divisor of ", 85085, " is the number ", 35

     "   One proper divisor of ", 85085, " is the number ", 55

     "   One proper divisor of ", 85085, " is the number ", 65
```

" One proper divisor of ", 85085, " is the number ", 77

" One proper divisor of ", 85085, " is the number ", 85

" One proper divisor of ", 85085, " is the number ", 91

" One proper divisor of ", 85085, " is the number ", 119

" One proper divisor of ", 85085, " is the number ", 143

" One proper divisor of ", 85085, " is the number ", 187

" One proper divisor of ", 85085, " is the number ", 221

" One proper divisor of ", 85085, " is the number ", 385

" One proper divisor of ", 85085, " is the number ", 455

" One proper divisor of ", 85085, " is the number ", 595

" One proper divisor of ", 85085, " is the number ", 715

" One proper divisor of ", 85085, " is the number ", 935

" One proper divisor of ", 85085, " is the number ", 1001

" One proper divisor of ", 85085, " is the number ", 1105

" One proper divisor of ", 85085, " is the number ", 1309

" One proper divisor of ", 85085, " is the number ", 1547

" One proper divisor of ", 85085, " is the number ", 2431

" One proper divisor of ", 85085, " is the number ", 5005

" One proper divisor of ", 85085, " is the number ", 6545

" One proper divisor of ", 85085, " is the number ", 7735

" One proper divisor of ", 85085, " is the number ", 12155

" One proper divisor of ", 85085, " is the number ", 17017

" and that is all of them. "

" count of proper divisors is ", 31

note that 31 = $2^5 – 1$.

> ProperDivisors(5*7);

" Input is ", 35, " Begin calculation."

"   One proper divisor of ", 35, " is the number ", 1

"   One proper divisor of ", 35, " is the number ", 5

"   One proper divisor of ", 35, " is the number ", 7

" and that is all of them. "

" count of proper divisors is ", 3

so 5*7 = 35 is a semi prime and has 2 distinct prime divisors.

our relationship of note is with Mersenne numbers 3 = $2^2 - 1$

another example to hammer this relationship home.

b = 11*13*19.

> ProperDivisors(19*(11*13));

" Input is ", 2717, " Begin calculation."

"   One proper divisor of ", 2717, " is the number ", 1

"   One proper divisor of ", 2717, " is the number ", 11

"   One proper divisor of ", 2717, " is the number ", 13

"   One proper divisor of ", 2717, " is the number ", 19

"   One proper divisor of ", 2717, " is the number ", 143

"   One proper divisor of ", 2717, " is the number ", 209

"   One proper divisor of ", 2717, " is the number ", 247

" and that is all of them. "

" count of proper divisors is ", 7

we see that our count is 7 and $2^3 - 1$. And our number of primes in b is 3.

woo hoo. Insight by Matt C Anderson.

and just because appendix 2 is not long enough, we will do an example of 4 consecutive primes starting at 11, and find a count of 15 discrete divisors. Since 2^4 – 1 is 15 and those are Mersenne numbers.

b = 11*13*17*19

```
> ProperDivisors(17*(11*13)*19);

print(`output redirected...`); # input placeholder

              " Input is ", 46189, " Begin calculation."

        "   One proper divisor of ", 46189, " is the number ", 1

        "   One proper divisor of ", 46189, " is the number ", 11

        "   One proper divisor of ", 46189, " is the number ", 13

        "   One proper divisor of ", 46189, " is the number ", 17

        "   One proper divisor of ", 46189, " is the number ", 19

        "   One proper divisor of ", 46189, " is the number ", 143

        "   One proper divisor of ", 46189, " is the number ", 187

        "   One proper divisor of ", 46189, " is the number ", 209

        "   One proper divisor of ", 46189, " is the number ", 221

        "   One proper divisor of ", 46189, " is the number ", 247

        "   One proper divisor of ", 46189, " is the number ", 323

        "   One proper divisor of ", 46189, " is the number ", 2431

        "   One proper divisor of ", 46189, " is the number ", 2717

        "   One proper divisor of ", 46189, " is the number ", 3553

        "   One proper divisor of ", 46189, " is the number ", 4199

                    " and that is all of them.  "

                  " count of proper divisors is ", 15
```

fun to observe.

there it is.