Minimal elements for the base *b* representations of the primes which are > *b*

Introduction

A string *x* is a subsequence of another string *y*, if *x* can be obtained from *y* by deleting zero or more of the characters in *y*. For example, 514 is a substring of 251664. The empty string is a subsequence of every string.

Two strings *x* and *y* are comparable if either *x* is a substring of *y*, or *y* is a substring of *x*. A surprising result from formal language theory is that every set of pairwise incomparable strings is finite. This means that from any set of strings we can find its minimal elements.

A string *x* in a set of strings *S* is a minimal string if whenever *y* (an element of *S*) is a substring of *x*, we have *y* = *x*.

The set of all minimal strings of *S* is denoted *M*(*S*), the set *M*(*S*) must be finite! Even if *S* is an infinite set, such as the set of prime strings in decimal.

Although the set *M*(*S*) of minimal strings is necessarily finite, determining it explicitly for a given *S* can be a difficult computational problem. We use some numbertheoretic heuristics to compute $M(L_b)$, where L_b is the language of base-*b* representations of the prime numbers which are > *b*, for 2 ≤ *b* ≤ 16.

b	L_b
2	11, 101, 111, 1011, 1101, 10001, 10011, 10111, 11101, 11111, 100101, 101001, 101011, 101111, 110101, 111011, 111101, 1000011, 1000111, 1001001, 1001111, 1010011, 1011001, 1100001, 1100101, 1100111, 1101011, 1101101, 1110001, 1111111, 10000011, 10001001, 10001011, 10010101, 10010111, 10011101, 10100011, 10100111, 10101101, 10110011, 10110101, 10111111, 11000001, 11000101, 11000111, 11010011, 11011111, 11100011, 11100101, 11101001, 11101111, 11110001, 11111011, ...
3	12, 21, 102, 111, 122, 201, 212, 1002, 1011, 1101, 1112, 1121, 1202, 1222, 2012, 2021, 2111, 2122, 2201, 2221, 10002, 10022, 10121, 10202, 10211, 10222, 11001, 11012, 11201, 11212, 12002, 12011, 12112, 12121, 12211, 20001, 20012, 20102, 20122, 20201, 21002, 21011, 21022, 21101, 21211, 22021, 22102, 22111, 22122, 22212, 22221, 100022, ...
4	11, 13, 23, 31, 101, 103, 113, 131, 133, 211, 221, 223, 233, 311, 323, 331,

	1003, 1013, 1021, 1033, 1103, 1121, 1201, 1211, 1213, 1223, 1231, 1301, 1333, 2003, 2021, 2023, 2111, 2113, 2131, 2203, 2213, 2231, 2303, 2311, 2333, 3001, 3011, 3013, 3103, 3133, 3203, 3211, 3221, 3233, 3301, 3323, ...
5	12, 21, 23, 32, 34, 43, 104, 111, 122, 131, 133, 142, 203, 214, 221, 232, 241, 243, 304, 313, 324, 342, 401, 403, 412, 414, 423, 1002, 1011, 1022, 1024, 1044, 1101, 1112, 1123, 1132, 1143, 1204, 1211, 1231, 1233, 1242, 1244, 1321, 1343, 1402, 1404, 1413, 1424, 1431, 2001, ...
6	11, 15, 21, 25, 31, 35, 45, 51, 101, 105, 111, 115, 125, 135, 141, 151, 155, 201, 211, 215, 225, 241, 245, 251, 255, 301, 305, 331, 335, 345, 351, 405, 411, 421, 431, 435, 445, 455, 501, 515, 521, 525, 531, 551, 1011, 1015, 1021, 1025, 1035, 1041, 1055, ...
7	14, 16, 23, 25, 32, 41, 43, 52, 56, 61, 65, 104, 113, 115, 124, 131, 133, 142, 146, 155, 166, 203, 205, 212, 214, 221, 241, 245, 254, 256, 302, 304, 313, 322, 326, 335, 344, 346, 362, 364, 401, 403, 421, 436, 443, 445, 452, 461, 463, 506, ...
8	13, 15, 21, 23, 27, 35, 37, 45, 51, 53, 57, 65, 73, 75, 103, 107, 111, 117, 123, 131, 141, 145, 147, 153, 155, 161, 177, 203, 211, 213, 225, 227, 235, 243, 247, 255, 263, 265, 277, 301, 305, 307, 323, 337, 343, 345, 351, 357, 361, 373, ...
9	12, 14, 18, 21, 25, 32, 34, 41, 45, 47, 52, 58, 65, 67, 74, 78, 81, 87, 102, 108, 117, 122, 124, 128, 131, 135, 151, 155, 162, 164, 175, 177, 184, 201, 205, 212, 218, 221, 232, 234, 238, 241, 254, 267, 272, 274, 278, 285, 287, 308, ...
10	11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, ...
11	12, 16, 18, 21, 27, 29, 34, 38, 3A, 43, 49, 54, 56, 61, 65, 67, 72, 76, 81, 89, 92, 94, 98, 9A, A3, 106, 10A, 115, 117, 126, 128, 133, 139, 142, 148, 153, 155, 164, 166, 16A, 171, 182, 193, 197, 199, 1A2, 1A8, 1AA, 209, ...
12	11, 15, 17, 1B, 25, 27, 31, 35, 37, 3B, 45, 4B, 51, 57, 5B, 61, 67, 6B, 75, 81, 85, 87, 8B, 91, 95, A7, AB, B5, B7, 105, 107, 111, 117, 11B, 125, 12B, 131, 13B, 141, 145, 147, 157, 167, 16B, 171, 175, 17B, 181, 18B, ...
13	14, 16, 1A, 23, 25, 2B, 32, 34, 38, 41, 47, 49, 52, 56, 58, 61, 65, 6B, 76, 7A, 7C, 83, 85, 89, 9A, A1, A7, A9, B6, B8, C1, C7, CB, 104, 10A, 10C, 119, 11B, 122, 124, 133, 142, 146, 148, 14C, 155, 157, 164, ...
14	13, 15, 19, 21, 23, 29, 2D, 31, 35, 3B, 43, 45, 4B, 51, 53, 59, 5D, 65, 6D, 73, 75, 79, 7B, 81, 91, 95, 9B, 9D, A9, AB, B3, B9, BD, C5, CB, CD, D9, DB, 101, 103, 111, 11D, 123, 125, 129, 131, 133, 13D, ...
15	12, 14, 18, 1E, 21, 27, 2B, 2D, 32, 38, 3E, 41, 47, 4B, 4D, 54, 58, 5E, 67, 6B, 6D, 72, 74, 78, 87, 8B, 92, 94, 9E, A1, A7, AD, B2, B8, BE, C1, CB, CD, D2, D4, E1, ED, 102, 104, 108, 10E, 111, 11B, ...
16	11, 13, 17, 1D, 1F, 25, 29, 2B, 2F, 35, 3B, 3D, 43, 47, 49, 4F, 53, 59, 61, 65, 67, 6B, 6D, 71, 7F, 83, 89, 8B, 95, 97, 9D, A3, A7, AD, B3, B5, BF, C1, C5, C7, D3, DF, E3, E5, E9, EF, F1, FB, ...

18	11, 15, 1B, 1D, 21, 25, 27, 2B, 2H, 35, 37, 3D, 3H, 41, 47, 4B, 4H, 57, 5B, 5D, 5H, 61, 65, 71, 75, 7B, 7D, 85, 87, 8D, 91, 95, 9B, 9H, A1, AB, AD, AH, B1, BD, C7, CB, CD, CH, D5, D7, DH, ...
20	13, 19, 1B, 1H, 21, 23, 27, 2D, 2J, 31, 37, 3B, 3D, 3J, 43, 49, 4H, 51, 53, 57, 59, 5D, 67, 6B, 6H, 6J, 79, 7B, 7H, 83, 87, 8D, 8J, 91, 9B, 9D, 9H, 9J, AB, B3, B7, B9, BD, BJ, C1, CB, ...
24	15, 17, 1D, 1H, 1J, 1N, 25, 2B, 2D, 2J, 2N, 31, 37, 3B, 3H, 41, 45, 47, 4B, 4D, 4H, 57, 5B, 5H, 5J, 65, 67, 6D, 6J, 6N, 75, 7B, 7D, 7N, 81, 85, 87, 8J, 97, 9B, 9D, 9H, 9N, A1, AB, ...
30	11, 17, 1B, 1D, 1H, 1N, 1T, 21, 27, 2B, 2D, 2J, 2N, 2T, 37, 3B, 3D, 3H, 3J, 3N, 47, 4B, 4H, 4J, 4T, 51, 57, 5D, 5H, 5N, 5T, 61, 6B, 6D, 6H, 6J, 71, 7D, 7H, 7J, 7N, 7T, 81, 8B, ...
32	15, 19, 1B, 1F, 1L, 1R, 1T, 23, 27, 29, 2F, 2J, 2P, 31, 35, 37, 3B, 3D, 3H, 3V, 43, 49, 4B, 4L, 4N, 4T, 53, 57, 5D, 5J, 5L, 5V, 61, 65, 67, 6J, 6V, 73, 75, 79, 7F, 7H, 7R, ...
36	11, 15, 17, 1B, 1H, 1N, 1P, 1V, 1Z, 21, 27, 2B, 2H, 2P, 2T, 2V, 2Z, 31, 35, 3J, 3N, 3T, 3V, 45, 47, 4D, 4J, 4N, 4T, 4Z, 51, 5B, 5D, 5H, 5J, 5V, 67, 6B, 6D, 6H, 6N, 6P, 6Z, ...

The primes in $M(L_b)$ are called **minimal prime base *b*** in this article, although this name should be used for L_b is the language of base-*b* representations of the prime numbers, where primes > *b* is not required, this problem an extension of the original minimal prime problem to include the generalized Sierpinski conjecture base *b* and the generalized Riesel conjecture base *b*, for all *k*-values < *b*. For example, 149 is not minimal prime in decimal, because 19 is prime and 19 is

Problems about the digits of prime numbers have a long history, and many of them are still unsolved. For example, are there infinitely many primes, all of whose base-10 digits are 1? Currently, there are only five such “repunits” known, corresponding to $(10^p - 1)/9$ for $p \in \{2, 19, 23, 317, 1031\}$. It seems likely that four more are given by $p \in \{49081, 86453, 109297, 270343\}$, but this has not yet been rigorously proven. This problem also exists for other bases, e.g. for base 12, there are only nine proven such numbers, corresponding to $(12^p - 1)/11$ for $p \in \{2, 3, 5, 19, 97, 109, 317, 353, 701\}$. It seems likely that five more are given by $p \in \{9739, 14951, 37573, 46889, 769543\}$, but this has not yet been rigorously proven. However, for some bases there exists no such primes, these bases are 9, 25, 32, 49, 64, 81, ... OEIS A096059

Table

In the “$max(x, x \in L_b)$” column, xy^nz means *xyyy...yyyz* with *n* *y*’s (the *n*-value is written in decimal), not *y* to the *n*th power.

b	$\|M(L_b)\|$	$max(x, x \in L_b)$	$max(\|x\|, x \in L_b)$	Decimal form of $max(x, x \in L_b)$

2	1	11	2	3
3	3	111	3	13
4	5	221	3	41
5	22	$10^{93}13$	96	$5^{95}+8$
6	11	40041	5	5209
7**	≥71	$3^{16}1$	17	$\frac{7^{17}-5}{2}$
8	75	$4^{220}7$	221	$\frac{4\cdot 8^{221}+17}{7}$
9**	≥139	$76^{329}2$	331	$\frac{31\cdot 9^{330}-19}{4}$
10	77	$50^{28}27$	31	$5\cdot 10^{30}+27$

** I have not proved these bases, these are the largest elements in $M(L_b)$ known to me, and they are just the lower bounds.

Lists

base 2

11

base 3

12, 21, 111

base 4

11, 13, 23, 31, 221

base 5

12, 21, 23, 32, 34, 43, 104, 111, 131, 133, 313, 401, 414, 3101, 10103, 14444, 30301, 33001, 33331, 44441, 300031,
100000000000000000000000000000000000000000000000000000000000000000000000000000000
000000000000013

base 6

11, 15, 21, 25, 31, 35, 45, 51, 4401, 4441, 40041

base 7 (not proved, only checked to the prime 5100000001)

14, 16, 23, 25, 32, 41, 43, 52, 56, 61, 65, 113, 115, 131, 133, 155, 212, 221, 304, 313, 335, 344, 346, 364, 445, 515, 533, 535, 544, 551, 553, 1022, 1051, 1112, 1202, 1211, 1222, 2111, 3031, 3055, 3334, 3503, 3505, 3545, 4504, 4555, 5011, 5455, 5545, 5554, 6034, 6634, 11111, 11201, 30011, 30101, 31001, 31111, 33001, 33311, 35555, 40054, 100121, 150001, 300053, 351101, 531101, 1100021, 33333301, 5100000001, ..., 33333333333333331, ...

base 8

13, 15, 21, 23, 27, 35, 37, 45, 51, 53, 57, 65, 73, 75, 107, 111, 117, 141, 147, 161, 177, 225, 255, 301, 343, 361, 401, 407, 417, 431, 433, 463, 467, 471, 631, 643, 661, 667, 701, 711, 717, 747, 767, 3331, 3411, 4043, 4443, 4611, 5205, 6007, 6101, 6441, 6477, 6707, 6777, 7461, 7641, 47777, 60171, 60411, 60741, 444641, 500025, 505525, 3344441, 4444477, 5500525, 5550525, 55555025, 444444441, 744444441, 77774444441, 7777777777771, 555555555555525,
444444444444444444444444444444444444444444444444444444444444444444444444444444444
444444444444444444444444444444444444444444444444444444444444444444444444444444444
44444444444444444444444444444444444444444444444444444444447

base 9 (not proved, only checked to the prime 8333333335)

12, 14, 18, 21, 25, 32, 34, 41, 45, 47, 52, 58, 65, 67, 74, 78, 81, 87, 117, 131, 135, 151, 155, 175, 177, 238, 272, 308, 315, 331, 337, 355, 371, 375, 377, 438, 504, 515, 517, 531, 537, 557, 564, 601, 638, 661, 702, 711, 722, 735, 737, 751, 755, 757, 771, 805, 838, 1011, 1015, 1101, 1701, 2027, 2207, 3017, 3057, 3101, 3501, 3561, 3611, 3688, 3868, 5035, 5051, 5071, 5101, 5501, 5554, 5705, 5707, 7017, 7075, 7105, 7301, 8535, 8544, 8555, 8854, 20777, 22227, 22777, 30161, 33388, 50161, 50611, 53335, 55111, 55535, 55551, 57061, 57775, 70631, 71007, 77207, 100037, 100071, 100761, 105007, 270707, 301111, 305111, 333035, 333385, 333835, 338885, 350007, 500075, 530005, 555611, 631111, 720707, 2770007, 3030335, 7776662, 30300005, 30333335, 38333335, 51116111, 70000361, 300030005, 300033305, 351111111, 1300000007, 5161111111, 8333333335, ..., 300000000035, ..., 544444444444, ..., 2000000000007, ...,
56111111111111111111111111111111111111, ...,
766666666666666666666666666666666666666666666666666666666666666666666666666666666
666666666666666666666666666666666666666666666666666666666666666666666666666666666
666666666666666666666666666666666666666666666666666666666666666666666666666666666
666666666666666666666666666666666666666666666666666666666666666666666666666666666
6666662, ...

base 10

11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 227, 251, 257, 277, 281, 349, 409, 449, 499, 521, 557, 577, 587, 727, 757, 787, 821, 827, 857, 877, 881, 887, 991, 2087, 2221, 5051, 5081, 5501, 5581, 5801, 5851, 6469, 6949, 8501, 9001, 9049, 9221, 9551, 9649, 9851, 9949, 20021, 20201, 50207, 60649, 80051, 666649, 946669, 5200007, 22000001, 60000049, 66000049, 66600049, 80555551, 555555555551, 5000000000000000000000000000027

Proof

As previously mentioned, in practice to compute $M(L_b)$ one works with an underapproximation M of $M(L_b)$ and an overapproximation L of $L_b - sup(M)$. One then refines such approximations until $L = \varnothing$ from which it follows that $M = M(L_b)$.

For the initial approximation, note that every minimal prime in base b with at least 4 digits is of the form xY^*z, where $x \in \{x \mid x$ is base-b digit, $x \neq 0\}$, $z \in \{z \mid z$ is base-b digit, $gcd(z,b) = 1\}$, and $Y_{xz} = \{y \mid xy, xz, yz, xyz$ are all composites$\}$.

Making use of this, our algorithm sets M to be the set of base-b representations of the minimal primes with at most 3 digits (which can be found simply by brute force) and L to be $\bigcup_{x,z}(xY^*z)$ as described above.

All remaining minimal primes are members of L, so to find them we explore the families in L. During this process, each family will be decomposed into possibly multiple other families. For example, a simple way of exploring the family xY^*z where $Y = \{y_1, . . . , y_n\}$ is to decompose it into the families xY ∗
y1z, . . . , xY ∗
ynz. If the smallest member (say xyiz) of
any such family happens to be prime, it can be added to M and the family xY ∗
yiz removed
from consideration. Furthermore, once M has been updated it may be possible to simplify some families in L. In this case, xY ∗
yjz (for j 6= i) can be simplified to x(Y − {yi})
∗
yjz since
no minimal prime contains xyiz as a proper subword.

For any given base b, we find all (x,z) digits-pair such that $x \neq 0$ and $gcd(z,b) = 1$, and find the corresponding sets Y^*, see below.

Bold for minimal primes in base b, i.e. elements of the set $M(L_b)$

base 2

The possible (first digit,last digit) for an element with >=3 digits in the minimal set of the strings for primes with at least two digits are:

(1,1)

* Case (1,1):

** **11** is prime, and thus the only minimal prime in this family.

base 3

The possible (first digit,last digit) for an element with >=3 digits in the minimal set of the strings for primes with at least two digits are:

(1,1), (1,2), (2,1), (2,2)

* Case (1,1):

** Since 12, 21, **111** are primes, we only need to consider the family 1{0}1 (since any digits 1, 2 between them will produce smaller primes)

*** All numbers of the form 1{0}1 are divisible by 2, thus cannot be prime.

* Case (1,2):

** **12** is prime, and thus the only minimal prime in this family.

* Case (2,1):

** **21** is prime, and thus the only minimal prime in this family.

* Case (2,2):

** Since 21, 12 are primes, we only need to consider the family 2{0,2}2 (since any digits 1 between them will produce smaller primes)

*** All numbers of the form 2{0,2}2 are divisible by 2, thus cannot be prime.

base 4

The possible (first digit,last digit) for an element with >=3 digits in the minimal set of the strings for primes with at least two digits are:

(1,1), (1,3), (2,1), (2,3), (3,1), (3,3)

* Case (1,1):

** **11** is prime, and thus the only minimal prime in this family.

* Case (1,3):

** **13** is prime, and thus the only minimal prime in this family.

* Case (2,1):

** Since 23, 11, 31, **221** are primes, we only need to consider the family 2{0}1 (since any digits 1, 2, 3 between them will produce smaller primes)

*** All numbers of the form 2{0}1 are divisible by 3, thus cannot be prime.

* Case (2,3):

** **23** is prime, and thus the only minimal prime in this family.

* Case (3,1):

** **31** is prime, and thus the only minimal prime in this family.

* Case (3,3):

** Since 31, 13, 23 are primes, we only need to consider the family 3{0,3}3 (since any digits 1, 2 between them will produce smaller primes)

*** All numbers of the form 3{0,3}3 are divisible by 3, thus cannot be prime.

base 5

The possible (first digit,last digit) for an element with >=3 digits in the minimal set of the strings for primes with at least two digits are:

(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)

* Case (1,1):

** Since 12, 21, **111**, **131** are primes, we only need to consider the family 1{0,4}1 (since any digits 1, 2, 3 between them will produce smaller primes)

*** All numbers of the form 1{0,4}1 are divisible by 2, thus cannot be prime.

* Case (1,2):

** **12** is prime, and thus the only minimal prime in this family.

* Case (1,3):

** Since 12, 23, 43, **133** are primes, we only need to consider the family 1{0,1}3 (since any digits 2, 3, 4 between them will produce smaller primes)

*** Since 111 is prime, we only need to consider the families 1{0}3 and 1{0}1{0}3 (since any digit combo 11 between (1,3) will produce smaller primes)

**** All numbers of the form 1{0}3 are divisible by 2, thus cannot be prime.

**** For the 1{0}1{0}3 family, since **10103** is prime, we only need to consider the families 1{0}13 and 11{0}3 (since any digit combo 010 between (1,3) will produce smaller primes)

***** The smallest prime of the form 1{0}13 is **100000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000013**, which can be written as $10_{93}13$ and equal the prime 5^95+8

***** All numbers of the form 11{0}3 are divisible by 3, thus cannot be prime.

* Case (1,4):

** Since 12, 34, **104** are primes, we only need to consider the families 1{1,4}4 (since any digits 0, 2, 3 between them will produce smaller primes)

*** Since 111, 414 are primes, we only need to consider the family 1{4}4 and 11{4}4 (since any digit combo 11 or 41 between them will produce smaller primes)

**** The smallest prime of the form 1{4}4 is **14444**.

**** All numbers of the form 11{4}4 are divisible by 2, thus cannot be prime.

* Case (2,1):

** **21** is prime, and thus the only minimal prime in this family.

* Case (2,2):

** Since 21, 23, 12, 32 are primes, we only need to consider the family 2{0,2,4}2 (since any digits 1, 3 between them will produce smaller primes)

*** All numbers of the form 2{0,2,4}2 are divisible by 2, thus cannot be prime.

* Case (2,3):

** **23** is prime, and thus the only minimal prime in this family.

* Case (2,4):

** Since 21, 23, 34 are primes, we only need to consider the family 2{0,2,4}4 (since any digits 1, 3 between them will produce smaller primes)

*** All numbers of the form 2{0,2,4}4 are divisible by 2, thus cannot be prime.

* Case (3,1):

** Since 32, 34, 21 are primes, we only need to consider the family 3{0,1,3}1 (since any digits 2, 4 between them will produce smaller primes)

*** Since 313, 111, 131, **3101** are primes, we only need to consider the families 3{0,3}1 and 3{0,3}11 (since any digit combo 10, 11, 13 between (3,1) will produce smaller primes)

**** For the 3{0,3}1 family, we can separate this family to four families:

***** For the 30{0,3}01 family, we have the prime **30301**, and the remain case is the family 30{0}01.

****** All numbers of the form 30{0}01 are divisible by 2, thus cannot be prime.

***** For the 30{0,3}31 family, note that there must be an even number of 3's between (30,31), or the result number will be divisible by 2 and cannot be prime.

****** Since 33331 is prime, any digit combo 33 between (30,31) will produce smaller primes.

******* Thus, the only possible prime is the smallest prime in the family 30{0}31, and this prime is **300031**.

***** For the 33{0,3}01 family, note that there must be an even number of 3's between (33,01), or the result number will be divisible by 2 and cannot be prime.

****** Since 33331 is prime, any digit combo 33 between (33,01) will produce smaller primes.

******* Thus, the only possible prime is the smallest prime in the family 33{0}01, and this prime is **33001**.

***** For the 33{0,3}31 family, we have the prime **33331**, and the remain case is the family 33{0}31.

****** All numbers of the form 33{0}31 are divisible by 2, thus cannot be prime.

* Case (3,2):

** **32** is prime, and thus the only minimal prime in this family.

* Case (3,3):

** Since 32, 34, 23, 43, **313** are primes, we only need to consider the family 3{0,3}3 (since any digits 1, 2, 4 between them will produce smaller primes)

*** All numbers of the form 3{0,3}3 are divisible by 3, thus cannot be prime.

* Case (3,4):

** **34** is prime, and thus the only minimal prime in this family.

* Case (4,1):

** Since 43, 21, **401** are primes, we only need to consider the family 4{1,4}1 (since any digits 0, 2, 3 between them will produce smaller primes)

*** Since 414, 111 are primes, we only need to consider the family 4{4}1 and 4{4}11 (since any digit combo 14 or 11 between them will produce smaller primes)

**** The smallest prime of the form 4{4}1 is **44441**.

**** All numbers of the form 4{4}11 are divisible by 2, thus cannot be prime.

* Case (4,2):

** Since 43, 12, 32 are primes, we only need to consider the family 4{0,2,4}2 (since any digits 1, 3 between them will produce smaller primes)

*** All numbers of the form 4{0,2,4}2 are divisible by 2, thus cannot be prime.

* Case (4,3):

** **43** is prime, and thus the only minimal prime in this family.

* Case (4,4):

** Since 43, 34, **414** are primes, we only need to consider the family 4{0,2,4}4 (since any digits 1, 3 between them will produce smaller primes)

*** All numbers of the form 4{0,2,4}4 are divisible by 2, thus cannot be prime.

base 6

The possible (first digit,last digit) for an element with >=3 digits in the minimal set of the strings for primes with at least two digits are:

(1,1), (1,5), (2,1), (2,5), (3,1), (3,5), (4,1), (4,5), (5,1), (5,5)

* Case (1,1):

** **11** is prime, and thus the only minimal prime in this family.

* Case (1,5):

** **15** is prime, and thus the only minimal prime in this family.

* Case (2,1):

** **21** is prime, and thus the only minimal prime in this family.

* Case (2,5):

** **25** is prime, and thus the only minimal prime in this family.

* Case (3,1):

** **31** is prime, and thus the only minimal prime in this family.

* Case (3,5):

** **35** is prime, and thus the only minimal prime in this family.

* Case (4,1):

** Since 45, 11, 21, 31, 51 are primes, we only need to consider the family 4{0,4}1 (since any digits 1, 2, 3, 5 between them will produce smaller primes)

*** Since **4401** and **4441** are primes, we only need to consider the families 4{0}1 and 4{0}41 (since any digits combo 40 and 44 between them will produce smaller primes)

**** All numbers of the form 4{0}1 are divisible by 5, thus cannot be prime.

**** The smallest prime of the form 4{0}41 is **40041**

* Case (4,5):

** **45** is prime, and thus the only minimal prime in this family.

* Case (5,1):

** **51** is prime, and thus the only minimal prime in this family.

* Case (5,5):

** Since 51, 15, 25, 35, 45 are primes, we only need to consider the family 5{0,5}5 (since any digits 1, 2, 3, 4 between them will produce smaller primes)

*** All numbers of the form 5{0,5}5 are divisible by 5, thus cannot be prime.

base 8

The possible (first digit,last digit) for an element with >=3 digits in the minimal set of the strings for primes with at least two digits are:

(1,1), (1,3), (1,5), (1,7), (2,1), (2,3), (2,5), (2,7), (3,1), (3,3), (3,5), (3,7), (4,1), (4,3), (4,5), (4,7), (5,1), (5,3), (5,5), (5,7), (6,1), (6,3), (6,5), (6,7), (7,1), (7,3), (7,5), (7,7)

* Case (1,1):

** Since 13, 15, 21, 51, **111**, **141**, **161** are primes, we only need to consider the family 1{0,7}1 (since any digits 1, 2, 3, 4, 5, 6 between them will produce smaller primes)

*** Since 107, 177, 701 are primes, we only need to consider the number 171 and the family 1{0}1 (since any digits combo 07, 70, 77 between them will produce smaller primes)

**** 171 is not prime.

**** All numbers of the form 1{0}1 factored as 10^n+1 = (2^n+1) * (4^n-2^n+1), thus cannot be prime.

* Case (1,3):

** **13** is prime, and thus the only minimal prime in this family.

* Case (1,5):

** **15** is prime, and thus the only minimal prime in this family.

* Case (1,7):

** Since 13, 15, 27, 37, 57, **107**, **117**, **147**, **177** are primes, we only need to consider the family 1{6}7 (since any digits 0, 1, 2, 3, 4, 5, 7 between them will produce smaller primes)

*** The smallest prime of the form 1{6}7 is 16667 (not minimal prime, since 667 is prime)

* Case (2,1):

** **21** is prime, and thus the only minimal prime in this family.

* Case (2,3):

** **23** is prime, and thus the only minimal prime in this family.

* Case (2,5):

** Since 21, 23, 27, 15, 35, 45, 65, 75, **225**, **255** are primes, we only need to consider the family 2{0}5 (since any digits 1, 2, 3, 4, 5, 6, 7 between them will produce smaller primes)

*** All numbers of the form 2{0}5 are divisible by 7, thus cannot be prime.

* Case (2,7):

** **27** is prime, and thus the only minimal prime in this family.

* Case (3,1):

** Since 35, 37, 21, 51, **301**, **361** are primes, we only need to consider the family 3{1,3,4}1 (since any digits 0, 2, 5, 6, 7 between them will produce smaller primes)

*** Since 13, 343, 111, 131, 141, 431, **3331**, **3411** are primes, we only need to consider the families 3{3}11, 33{1,4}1, 3{3,4}4{4}1 (since any digits combo 11, 13, 14, 33, 41, 43 between them will produce smaller primes)

**** All numbers of the form 3{3}11 are divisible by 3, thus cannot be prime.

**** For the 33{1,4}1 family, since 111 and 141 are primes, we only need to consider the families 33{4}1 and 33{4}11 (since any digits combo 11, 14 between them will produce smaller primes)

***** The smallest prime of the form 33{4}1 is **3344441**

***** All numbers of the form 33{4}11 are divisible by 301, thus cannot be prime.

**** For the 3{3,4}4{4}1 family, since 3331 and 3344441 are primes, we only need to consider the families 3{4}1, 3{4}31, 3{4}341, 3{4}3441, 3{4}34441 (since any digits combo 33 or 34444 between (3,1) will produce smaller primes)

***** All numbers of the form 3{4}1 are divisible by 31, thus cannot be prime.

***** Since 4443 is prime, we only need to consider the numbers 3431, 34431, 34341, 344341, 343441, 3443441, 3434441, 34434441 (since any digit combo 444 between (3,3{4}1) will produce smaller primes)

****** None of 3431, 34431, 34341, 344341, 343441, 3443441, 3434441, 34434441 are primes.

* Case (3,3):

** Since 35, 37, 13, 23, 53, 73, **343** are primes, we only need to consider the family 3{0,3,6}3 (since any digits 1, 2, 4, 5, 7 between them will produce smaller primes)

*** All numbers of the form 3{0,3,6}3 are divisible by 3, thus cannot be prime.

* Case (3,5):

** **35** is prime, and thus the only minimal prime in this family.

* Case (3,7):

** **37** is prime, and thus the only minimal prime in this family.

* Case (4,1):

** Since 45, 21, 51, **401**, **431**, **471** are primes, we only need to consider the family 4{1,4,6}1 (since any digits 0, 2, 3, 5, 7 between them will produce smaller primes)

*** Since 111, 141, 161, 661, **4611** are primes, we only need to consider the families 4{4}11, 4{4,6}4{1,4,6}1, 4{4}6{4}1 (since any digits combo 11, 14, 16, 61, 66 between them will produce smaller primes)

**** The smallest prime of the form 4{4}11 is 44444444444444411 (not minimal prime, since 444444441 is prime)

**** For the 4{4,6}4{1,4,6}1 family, we can separate this family to 4{4,6}41, 4{4,6}411, 4{4,6}461

**** For the 4{4,6}41 family, since 661 and 6441 are primes, we only need to consider the families 4{4}41 and 4{4}641 (since any digits combo 64 or 66 between (4,41) will produce smaller primes)

***** The smallest prime of the form 4{4}41 is **444444441**

***** The smallest prime of the form 4{4}641 is **444641**

**** For the 4{4,6}411 family, since 661 and 6441 are primes, we only need to consider the families 4{4}411 and 4{4}6411 (since any digits combo 64 or 66 between (4,411) will produce smaller primes)

***** The smallest prime of the form 4{4}411 is **444444441**

***** The smallest prime of the form 4{4}6411 is 4444444444444446411 (not minimal prime, since 444444441 and 444641 are primes)

**** For the 4{4,6}461 family, since 661 is prime, we only need to consider the family 4{4}461

***** The smallest prime of the form 4{4}461 is 4444444461 (not minimal prime, since 444444441 is prime)

**** For the 4{4}6{4}1 family, since 6441 is prime, we only need to consider the families 4{4}61 and 4{4}641 (since any digits combo 44 between (4{4}6,1) will produce smaller primes)

***** The smallest prime of the form 4{4}61 is 4444444461 (not minimal prime, since 444444441 is prime)

***** The smallest prime of the form 4{4}641 is **444641**

* Case (4,3):

** Since 45, 13, 23, 53, 73, **433**, **463** are primes, we only need to consider the family 4{0,4}3 (since any digits 1, 2, 3, 5, 6, 7 between them will produce smaller primes)

*** Since **4043** and **4443** are primes, we only need to consider the families 4{0}3 and 44{0}3 (since any digits combo 04, 44 between them will produce smaller primes)

**** All numbers of the form 4{0}3 are divisible by 7, thus cannot be prime.

**** All numbers of the form 44{0}3 are divisible by 3, thus cannot be prime.

* Case (4,5):

** **45** is prime, and thus the only minimal prime in this family.

* Case (4,7):

** Since 45, 27, 37, 57, **407**, **417**, **467** are primes, we only need to consider the family 4{4,7}7 (since any digits 0, 1, 2, 3, 5, 6 between them will produce smaller primes)

*** Since 747 is prime, we only need to consider the families 4{4}7, 4{4}77, 4{7}7, 44{7}7 (since any digits combo 74 between (4,7) will produce smaller primes)

**** The smallest prime of the form 4{4}7 is
**444444444444444444444444444444444444444444444444444444444444444444444444444444444
444444444444444444444444444444444444444444444444444444444444444444444444444444444
44444444444444444444444444444444444444444444444444444444447**, with 220 4's, which can be written as $4_{220}7$ and equal the prime (2^665+17)/7

**** The smallest prime of the form 4{4}77 is **4444477**

**** The smallest prime of the form 4{7}7 is **47777**

**** The smallest prime of the form 44{7}7 is
447777777777777777777777777777777777777777777777777777777777777777777777777777777
777777777777777777777777777777777777777777777777777777777777777777777777777777777
777777777777777777777777777777777777777777777777777777777777777777777777777777777
777777777777777777777777777777777777777777777777777777777777777777777777777777777
777777777777777777777777777777777777777777777777777777777777777777777777777777777
777777777777777777777777777777777777777777777777777777777777777777777777777777777
777777777777777777777777777777777777777777777777777777777777777777777777777777777
777777777777777777777777777777777777777777777777777777777777777777777777777777777
777777777777777777777777777777777777777777777777777777777777777777777777777777777

777777777777777777777777777777777777777777777777777777777777777777777777777777777777
77777777777777777777777777777777777777777777, with 851 7's, which can be written as 447_{851} and equal the prime 37*2^2553-1 (not minimal prime, since 47777 is prime)

* Case (5,1):

** **51** is prime, and thus the only minimal prime in this family.

* Case (5,3):

** **53** is prime, and thus the only minimal prime in this family.

* Case (5,5):

** Since 51, 53, 57, 15, 35, 45, 65, 75 are primes, we only need to consider the family 5{0,2,5}5 (since any digits 1, 3, 4, 6, 7 between them will produce smaller primes)

*** Since 225, 255, **5205** are primes, we only need to consider the families 5{0,5}5 and 5{0,5}25 (since any digits combo 20, 22, 25 between them will produce smaller primes)

**** All numbers of the form 5{0,5}5 are divisible by 5, thus cannot be prime.

**** For the 5{0,5}25 family, since **500025** and **505525** are primes, we only need to consider the number 500525 the families 5{5}25, 5{5}025, 5{5}0025, 5{5}0525, 5{5}00525, 5{5}05025 (since any digits combo 000, 055 between (5,25) will produce smaller primes)

***** 500525 is not prime.

***** The smallest prime of the form 5{5}25 is **555555555555525**

***** The smallest prime of the form 5{5}025 is **55555025**

***** The smallest prime of the form 5{5}0025 is
5555555555555555555555555555555555555555555555555555555555555555555555555555555555
5555555555555555555555555555555555555555555555555555555555555555555555555555555555
5555555555555555555550025 (not minimal prime, since 55555025 and 555555555555525 are primes)

***** The smallest prime of the form 5{5}0525 is **5550525**

***** The smallest prime of the form 5{5}00525 is **5500525**

***** The smallest prime of the form 5{5}05025 is 55555555555555555555555505025 (not minimal prime, since 5550525, 55555025, and 555555555555525 are primes)

* Case (5,7):

** **57** is prime, and thus the only minimal prime in this family.

* Case (6,1):

** Since 65, 21, 51, 631, 661 are primes, we only need to consider the family 6{0,1,4,7}1 (since any digits 2, 3, 5, 6 between them will produce smaller primes)

*** Since 111, 141, 401, 471, 701, 711, 6101, 6441 are primes, we only need to consider the families 6{0}0{0,1,4,7}1, 6{0,4}1{7}1, 6{0,7}4{1}1, 6{0,1,7}7{4,7}1 (since any digits combo 11, 14, 40, 47, 70, 71, 10, 44 between them will produce smaller primes)

**** For the 6{0}0{0,1,4,7}1 family, since 6007 is prime, we only need to consider the families 6{0}0{0,1,4}1 and 60{1,4,7}7{0,1,4,7}1 (since any digits combo 1007 between (6,1) will produce smaller primes)

***** For the 6{0}0{0,1,4}1 family, since 111, 141, 401, 6101, 6441, 60411 are primes, we only need to consider the families 6{0}1, 6{0}11, 6{0}41 (since any digits combo 10, 11, 14, 40, 41, 44 between (6{0}0,1) will produce smaller primes)

****** All numbers of the form 6{0}1 are divisible by 7, thus cannot be prime.

****** All numbers of the form 6{0}11 are divisible by 3, thus cannot be prime.

****** All numbers of the form 6{0}41 are divisible by 3, thus cannot be prime.

***** For the 60{1,4,7}7{0,1,4,7}1 family, since 701, 711, 60741 are primes, we only need to consider the family 60{1,4,7}7{7}1 (since any digits 0, 1, 4 between (60{1,4,7}7,1) will produce smaller primes)

***** Since 471, 60171 is prime, we only need to consider the family 60{7}1 (since any digits 1, 4 between (60,7{7}1) will produce smaller primes)

****** All numbers of the form 60{7}1 are divisible by 7, thus cannot be prime.

**** For the 6{0,4}1{7}1 family, since 417, 471 are primes, we only need to consider the families 6{0}1{7}1 and 6{0,4}11

***** For the 6{0}1{7}1 family, since 60171 is prime, and thus the only minimal prime in the family 6{0}1{7}1.

***** For the 6{0,4}11 family, since 401, 6441, 60411 are primes, we only need to consider the number 6411 and the family 6{0}11

****** 6411 is not prime.

****** All numbers of the form 6{0}11 are divisible by 3, thus cannot be prime.

**** For the 6{0,7}4{1}1 family, since 60411 is prime, we only need to consider the families 6{7}4{1}1 and 6{0,7}41

***** For the 6{7}4{1}1 family, since 111, 6777 are primes, we only need to consider the numbers 641, 6411, 6741, 67411, 67741, 677411

****** None of 641, 6411, 6741, 67411, 67741, 677411 are primes.

***** For the 6{0,7}41 family, since 701, 6777, 60741 are primes, we only need to consider the families 6{0}41 and the numbers 6741, 67741 (since any digits combo 07, 70, 777 between (6,41) will produce smaller primes)

****** All numbers of the form 6{0}41 are divisible by 3, thus cannot be prime.

****** Neither of 6741, 67741 are primes.

***** For the 6{0,1,7}7{4,7}1 family, since 747 is prime, we only need to consider the families 6{0,1,7}7{4}1, 6{0,1,7}7{7}1, 6{0,1,7}7{7}{4}1 (since any digits combo 47 between (6{0,1,7}7,1) will produce smaller primes)

****** For the 6{0,1,7}7{4}1 family, since 6441 is prime, we only need to consider the families 6{0,1,7}71 and 6{0,1,7}741 (since any digits combo 44 between (6{0,1,7}7,1) will produce smaller primes)

******* For the 6{0,1,7}71 family, since all numbers of the form 6{0,7}71 are divisible by 7 and cannot be prime, and 111 is prime (thus, any digits combo 11 between (6,71) will produce smaller primes), we only need to consider the family 6{0,7}1{0,7}71

******** Since 717 and 60171 are primes, we only need to consider the family 61{0,7}71 (since any digit combo 0, 7 between (6,1{0,7}71) will produce smaller primes)

********* Since 177 and 6101 are primes, we only need to consider the number 6171 (since any digit combo 0, 7 between (61,71) will produce smaller primes)

********** 6171 is not prime.

****** All numbers in the 6{0,1,7}7{7}1 or 6{0,1,7}7{7}{4}1 families are also in the 6{0,1,7}7{4}1 family, thus these two families cannot have more minimal primes.

* Case (6,3):

** Since 65, 13, 23, 53, 73, **643** are primes, we only need to consider the family 6{0,3,6}3 (since any digits 1, 2, 4, 5, 7 between them will produce smaller primes)

*** All numbers of the form 6{0,3,6}3 are divisible by 3, thus cannot be prime.

* Case (6,5):

** **65** is prime, and thus the only minimal prime in this family.

* Case (6,7):

** Since 65, 27, 37, 57, **667** are primes, we only need to consider the family 6{0,1,4,7}7 (since any digits 2, 3, 5, 6 between them will produce smaller primes)

*** Since 107, 117, 147, 177, 407, 417, 717, 747, **6007**, **6477**, **6707**, **6777** are primes, we only need to consider the families 60{1,4,7}7, 6{0}17, 6{0,4}4{4}7, 6{0}77 (since any digits combo 00, 10, 11, 14, 17, 40, 41, 47, 70, 71, 74, 77 between them will produce smaller primes)

**** All numbers of the form 6{0}17 or 6{0}77 are divisible by 3, thus cannot be prime.

**** For the 60{1,4,7}7 family, since 117, 147, 177, 417, 6477, 717, 747, 6777 are primes, we only need to consider the numbers 6017, 6047, 6077 and the family 60{4}7 (since any digit combo 11, 14, 17, 41, 47, 71, 74, 77 between (60,7) will produce smaller primes)

***** None of 6017, 6047, 6077 are primes.

**** All numbers of the form 60{4}7 are divisible by 21, thus cannot be prime.

**** For the 6{0,4}4{4}7 family, since 6007 and 407 are primes, we only need to consider the families 6{4}7 and 60{4}7 (since any digit combo 00, 40 between (6,4{4}7) will produce smaller primes)

***** All numbers of the form 6{4}7 are divisible by 3, 5, or 15, thus cannot be prime.

***** All numbers of the form 60{4}7 are divisible by 21, thus cannot be prime.

* Case (7,1):

** Since 73, 75, 21, 51, **701**, **711** are primes, we only need to consider the family 7{4,6,7}1 (since any digits 0, 1, 2, 3, 5 between them will produce smaller primes)

*** Since 747, 767, 471, 661, **7461**, **7641** are primes, we only need to consider the families 7{4,7}4{4}1, 7{7}61, 7{7}7{4,6,7}1 (since any digits combo 46, 47, 64, 66, 67 between them will produce smaller primes)

**** For the 7{4,7}4{4}1 family, since 747, 471 are primes, we only need to consider the family 7{7}{4}1 (since any digits combo 47 between (7,4{4}1) will produce smaller primes)

***** The smallest prime of the form 7{7}1 is **7777777777771**

***** The smallest prime of the form 7{7}41 is
777777777777777777777777777777777777777777777777777777777777777777777777777777741
(not minimal prime, since 7777777777771 is prime)

***** The smallest prime of the form 7{7}441 is
777777777777777777777777777777777777777777777777777777777777777777777777777777777
777441 (not minimal prime, since 7777777777771 is prime)

***** The smallest prime of the form 7{7}4441 is
777777777777777777777777777777777777777777777777777777777777777777777777777777777
777777777777777777777777777777777777777777777777777777777777777777777777777777777
777777777777777777777777777777777777777777777777777777777777777777777774441 (not minimal prime, since 7777777777771 is prime)

***** The smallest prime of the form 7{7}44441 is
7777777777777777777777777777777777777777777777777777777744441 (not minimal prime, since 7777777777771 is prime)

***** All numbers of the form 7{7}444441 are divisible by 7, thus cannot be prime.

***** The smallest prime of the form 7{7}4444441 is **77774444441**

****** Since this prime has just 4 7's, we only need to consider the families with <=3 7's

******* The smallest prime of the form 7{4}1 is **744444441**

******* All numbers of the form 77{4}1 are divisible by 5, thus cannot be prime.

******* The smallest prime of the form 777{4}1 is 7774444444444441 (not minimal prime, since 444444441 and 744444441 are primes)

* Case (7,3):

** **73** is prime, and thus the only minimal prime in this family.

* Case (7,5):

** **75** is prime, and thus the only minimal prime in this family.

* Case (7,7):

** Since 73, 75, 27, 37, 57, **717**, **747**, **767** are primes, we only need to consider the family 7{0,7}7 (since any digits 1, 2, 3, 4, 5, 6 between them will produce smaller primes)

*** All numbers of the form 7{0,7}7 are divisible by 7, thus cannot be prime.

base 10

The possible (first digit,last digit) for an element with >=3 digits in the minimal set of the strings for primes with at least two digits are:

(1,1), (1,3), (1,7), (1,9), (2,1), (2,3), (2,7), (2,9), (3,1), (3,3), (3,7), (3,9), (4,1), (4,3), (4,7), (4,9), (5,1), (5,3), (5,7), (5,9), (6,1), (6,3), (6,7), (6,9), (7,1), (7,3), (7,7), (7,9), (8,1), (8,3), (8,7), (8,9), (9,1), (9,3), (9,7), (9,9)

* Case (1,1):

** **11** is prime, and thus the only minimal prime in this family.

* Case (1,3):

** **13** is prime, and thus the only minimal prime in this family.

* Case (1,7):

** **17** is prime, and thus the only minimal prime in this family.

* Case (1,9):

** **19** is prime, and thus the only minimal prime in this family.

* Case (2,1):

** Since 23, 29, 11, 31, 41, 61, 71, **251**, **281** are primes, we only need to consider the family 2{0,2}1 (since any digits 1, 3, 4, 5, 6, 7, 8, 9 between them will produce smaller primes)

*** Since **2221** and **20201** are primes, we only need to consider the families 2{0}1, 2{0}21, 22{0}1 (since any digits combo 22 or 020 between them will produce smaller primes)

**** All numbers of the form 2{0}1 are divisible by 3, thus cannot be prime.

**** The smallest prime of the form 2{0}21 is **20021**

**** The smallest prime of the form 22{0}1 is **22000001**

* Case (2,3):

** **23** is prime, and thus the only minimal prime in this family.

* Case (2,7):

** Since 23, 29, 17, 37, 47, 67, 97 **227**, **257**, **277** are primes, we only need to consider the family 2{0,8}7 (since any digits 1, 2, 3, 4, 5, 6, 7, 9 between them will produce smaller primes)

*** Since 887 and **2087** are primes, we only need to consider the families 2{0}7 and 28{0}7 (since any digit combo 08 or 88 between them will produce smaller primes)

**** All numbers of the form 2{0}7 are divisible by 3, thus cannot be prime.

**** All numbers of the form 28{0}7 are divisible by 7, thus cannot be prime.

* Case (2,9):

** **29** is prime, and thus the only minimal prime in this family.

* Case (3,1):

** **31** is prime, and thus the only minimal prime in this family.

* Case (3,3):

** Since 31, 37, 13, 23, 43, 53, 73, 83 are primes, we only need to consider the family 3{0,3,6,9}3 (since any digits 1, 2, 4, 5, 7, 8 between them will produce smaller primes)

*** All numbers of the form 3{0,3,6,9}3 are divisible by 3, thus cannot be prime.

* Case (3,7):

** **37** is prime, and thus the only minimal prime in this family.

* Case (3,9):

** Since 31, 37, 19, 29, 59, 79, 89, **349** are primes, we only need to consider the family 3{0,3,6,9}9 (since any digits 1, 2, 4, 5, 7, 8 between them will produce smaller primes)

*** All numbers of the form 3{0,3,6,9}9 are divisible by 3, thus cannot be prime.

* Case (4,1):

** **41** is prime, and thus the only minimal prime in this family.

* Case (4,3):

** **43** is prime, and thus the only minimal prime in this family.

* Case (4,7):

** **47** is prime, and thus the only minimal prime in this family.

* Case (4,9):

** Since 41, 43, 47, 19, 29, 59, 79, 89, **409**, **449**, **499** are primes, we only need to consider the family 4{6}9 (since any digits 0, 1, 2, 3, 4, 5, 7, 8, 9 between them will produce smaller primes)

* Case (5,1):

** Since 53, 59, 11, 31, 41, 61, 71, **521** are primes, we only need to consider the family 5{0,5,8}1 (since any digits 1, 2, 3, 4, 6, 7, 9 between them will produce smaller primes)

*** Since 881 is prime, we only need to consider the families 5{0,5}1 and 5{0,5}8{0,5}1 (since any digit combo 88 between them will produce smaller primes)

**** For the 5{0,5}1 family, since **5051** and **5501** are primes, we only need to consider the families 5{0}1 and 5{5}1 (since any digit combo 05 or 50 between them will produce smaller primes)

***** All numbers of the form 5{0}1 are divisible by 3, thus cannot be prime.

***** The smallest prime of the form 5{5}1 is **555555555551**

**** For the 5{0,5}8{0,5}1 family, since **5081**, **5581**, **5801**, **5851** are primes, we only need to consider the number 581

***** 581 is not prime.

* Case (5,3):

** **53** is prime, and thus the only minimal prime in this family.

* Case (5,7):

** Since 53, 59, 17, 37, 47, 67, 97, **557**, **577**, **587** are primes, we only need to consider the family 5{0,2}7 (since any digits 1, 3, 4, 5, 6, 7, 8, 9 between them will produce smaller primes)

*** Since 227 and **50207** are primes, we only need to consider the families 5{0}7, 5{0}27, 52{0}7 (since any digits combo 22 or 020 between them will produce smaller primes)

**** All numbers of the form 5{0}7 are divisible by 3, thus cannot be prime.

**** The smallest prime of the form 5{0}27 is **5000000000000000000000000000027**

**** The smallest prime of the form 52{0}7 is **5200007**

* Case (5,9):

** **59** is prime, and thus the only minimal prime in this family.

* Case (6,1):

** **61** is prime, and thus the only minimal prime in this family.

* Case (6,3):

** Since 61, 67, 13, 23, 43, 53, 73, 83 are primes, we only need to consider the family 6{0,3,6,9}3 (since any digits 1, 2, 4, 5, 7, 8 between them will produce smaller primes)

*** All numbers of the form 6{0,3,6,9}3 are divisible by 3, thus cannot be prime.

* Case (6,7):

** **67** is prime, and thus the only minimal prime in this family.

* Case (6,9):

** Since 61, 67, 19, 29, 59, 79, 89 are primes, we only need to consider the family 6{0,3,4,6,9}9 (since any digits 1, 2, 5, 7, 8 between them will produce smaller primes)

*** Since 449 is prime, we only need to consider the families 6{0,3,6,9}9 and 6{0,3,6,9}4{0,3,6,9}9 (since any digit combo 44 between them will produce smaller primes)

**** All numbers of the form 6{0,3,6,9}9 are divisible by 3, thus cannot be prime.

**** For the 6{0,3,6,9}4{0,3,6,9}9 family, since 409, 43, **6469**, 499 are primes, we only need to consider the family 6{0,3,6,9}49

***** Since 349, **6949** are primes, we only need to consider the family 6{0,6}49

****** Since **60649** is prime, we only need to consider the family 6{6}{0}49 (since any digits combo 06 between {6,49} will produce smaller primes)

******* The smallest prime of the form 6{6}49 is **666649**

******** Since this prime has just 4 6's, we only need to consider the families with <=3 6's

********* The smallest prime of the form 6{0}49 is **60000049**

********* The smallest prime of the form 66{0}49 is **66000049**

********* The smallest prime of the form 666{0}49 is **66600049**

* Case (7,1):

** **71** is prime, and thus the only minimal prime in this family.

* Case (7,3):

** **73** is prime, and thus the only minimal prime in this family.

* Case (7,7):

** Since 71, 73, 79, 17, 37, 47, 67, 97, **727**, **757**, **787** are primes, we only need to consider the family 7{0,7}7 (since any digits 1, 2, 3, 4, 5, 6, 8, 9 between them will produce smaller primes)

*** All numbers of the form 7{0,7}7 are divisible by 7, thus cannot be prime.

* Case (7,9):

** **79** is prime, and thus the only minimal prime in this family.

* Case (8,1):

** Since 83, 89, 11, 31, 41, 61, 71, **821**, **881** are primes, we only need to consider the family 8{0,5}1 (since any digits 1, 2, 3, 4, 6, 7, 8, 9 between them will produce smaller primes)

*** Since **8501** is prime, we only need to consider the family 8{0}{5}1 (since any digits combo 50 between them will produce smaller primes)

**** Since **80051** is prime, we only need to consider the families 8{0}1, 8{5}1, 80{5}1 (since any digits combo 005 between them will produce smaller primes)

***** All numbers of the form 8{0}1 are divisible by 3, thus cannot be prime.

***** The smallest prime of the form 8{5}1 is 8555555555555555555551 (not minimal prime, since 555555555551 is prime)

***** The smallest prime of the form 80{5}1 is **80555551**

* Case (8,3):

** **83** is prime, and thus the only minimal prime in this family.

* Case (8,7):

** Since 83, 89, 17, 37, 47, 67, 97, **827**, **857**, **877**, **887** are primes, we only need to consider the family 8{0}7 (since any digits 1, 2, 3, 4, 5, 6, 7, 8, 9 between them will produce smaller primes)

*** All numbers of the form 8{0}7 are divisible by 3, thus cannot be prime.

* Case (8,9):

** **89** is prime, and thus the only minimal prime in this family.

* Case (9,1):

** Since 97, 11, 31, 41, 61, 71, **991** are primes, we only need to consider the family 9{0,2,5,8}1 (since any digits 1, 3, 4, 6, 7, 9 between them will produce smaller primes)

*** Since 251, 281, 521, 821, 881, **9001**, **9221**, **9551**, **9851** are primes, we only need to consider the families 9{2,5,8}0{2,5,8}1, 9{0}2{0}1, 9{0}5{0,8}1, 9{0,5}8{0}1 (since any digits combo 00, 22, 25, 28, 52, 55, 82, 85, 88 between them will produce smaller primes)

**** For the 9{2,5,8}0{2,5,8}1 family, since any digits combo 22, 25, 28, 52, 55, 82, 85, 88 between (9,1) will produce smaller primes, we only need to consider the numbers 901, 9021, 9051, 9081, 9201, 9501, 9801, 90581, 95081, 95801

***** 95801 is the only prime among 901, 9021, 9051, 9081, 9201, 9501, 9801, 90581, 95081, 95801, but it is not minimal prime since 5801 is prime.

**** For the 9{0}2{0}1 family, since 9001 is prime, we only need to consider the numbers 921, 9201, 9021

***** None of 921, 9201, 9021 are primes.

**** For the 9{0}5{0,8}1 family, since 9001 and 881 are primes, we only need to consider the numbers 951, 9051, 9501, 9581, 90581, 95081, 95801

***** 95801 is the only prime among 951, 9051, 9501, 9581, 90581, 95081, 95801, but it is not minimal prime since 5801 is prime.

**** For the 9{0,5}8{0}1 family, since 9001 and 5581 are primes, we only need to consider the numbers 981, 9081, 9581, 9801, 90581, 95081, 95801

***** 95801 is the only prime among 981, 9081, 9581, 9801, 90581, 95081, 95801, but it is not minimal prime since 5801 is prime.

* Case (9,3):

** Since 97, 13, 23, 43, 53, 73, 83 are primes, we only need to consider the family 9{0,3,6,9}3 (since any digits 1, 2, 4, 5, 7, 8 between them will produce smaller primes)

*** All numbers of the form 9{0,3,6,9}3 are divisible by 3, thus cannot be prime.

* Case (9,7):

** **97** is prime, and thus the only minimal prime in this family.

* Case (9,9):

** Since 97, 19, 29, 59, 79, 89 are primes, we only need to consider the family 9{0,3,4,6,9}9 (since any digits 1, 2, 5, 7, 8 between them will produce smaller primes)

*** Since 449 is prime, we only need to consider the families 9{0,3,6,9}9 and 9{0,3,6,9}4{0,3,6,9}9 (since any digit combo 44 between them will produce smaller primes)

**** All numbers of the form 9{0,3,6,9}9 are divisible by 3, thus cannot be prime.

**** For the 9{0,3,6,9}4{0,3,6,9}9 family, since **9049**, 349, **9649**, **9949** are primes, we only need to consider the family 94{0,3,6,9}9

***** Since 409, 43, 499 are primes, we only need to consider the family 94{6}9 (since any digits 0, 3, 9 between (94,9) will produce smaller primes)

****** The smallest prime of the form 94{6}9 is **946669**